Lezione 6

$\mathbb{R}^3$ $\qquad \Omega^1(\mathbb{R}^3) \cong \mathfrak{X}(\mathbb{R}^3) \qquad \Omega^2(\mathbb{R}^3) \cong \mathfrak{X}(\mathbb{R}^3)$

γ curva (1-sottovarietà di $\mathbb{R}^3$)

$\omega \in \Omega^1_c(\mathbb{R}^3) \qquad \omega = f\,dx + g\,dy + h\,dz$

$$\int_\gamma \omega = \int_\gamma X \cdot \mathbf{t} \qquad X = \begin{pmatrix} f \\ g \\ h \end{pmatrix}$$

$\mathbf{t}$ versore tangente

dim:

$\gamma(t) \qquad \gamma : I \to \mathbb{R}^3 \qquad I \subseteq \mathbb{R}$ intervallo

$$\gamma(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} \qquad x : I \to \mathbb{R}$$

$$\int_\gamma \omega = \int_\gamma f\,dx + g\,dy + h\,dz := \int_I f \frac{\partial x}{\partial t} dt + g \frac{\partial y}{\partial t} dt + h \frac{\partial z}{\partial t} dt$$

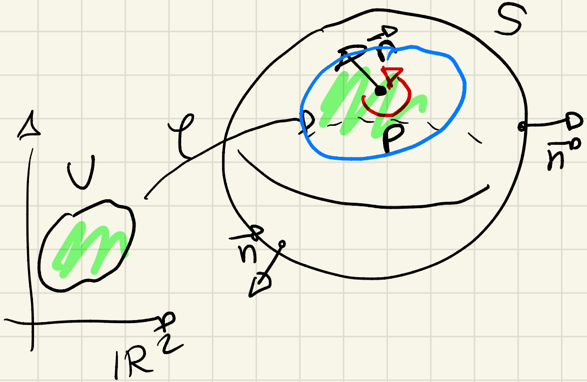
$$= \int_I \left(f\dot{x} + g\dot{y} + h\dot{z} \right) dt =$$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \dot{\gamma}(t) = \|\dot{\gamma}\| \cdot \mathbf{t}$$

$$= \int_I \langle X, \dot{\gamma} \rangle \, dt$$

$$= \int_I \langle X, \mathbf{t} \rangle \boxed{\|\dot{\gamma}\| \, dt} = \int_\gamma \langle X, \mathbf{t} \rangle$$

$\omega \in \Omega^2(\mathbb{R}^3)$ $\quad \omega = f\,dy \wedge dz + g\,dz \wedge dx + h\,dx \wedge dy$



$$X = \begin{pmatrix} f \\ g \\ h \end{pmatrix} \in \mathfrak{X}(\mathbb{R}^3)$$

$$\int_S \omega = \int_S X \cdot \vec{n}$$

$\vec{n}$ versore normale

$X \cdot \vec{n}$

$\langle X, \vec{n} \rangle$

Localmente S è del tipo parametrizzazione loc.

$U \subseteq \mathbb{R}^2 \qquad U \xrightarrow{\varphi} \mathbb{R}^3$

u, v

$$\varphi(u,v) = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix}$$

$$\omega = f\,dy \wedge dz + g\,dz \wedge dx + h\,dx \wedge dy$$

$$\int_{\varphi(U)} \omega := \int_U \varphi^*(\omega) = \int_U f\,dy \wedge dz + g\,dz \wedge dx + h\,dx \wedge dy =$$

$$x, y, z : U \to \mathbb{R} \qquad dx = \frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv$$

$$J = \begin{pmatrix} \overset{x_u}{\dfrac{\partial x}{\partial u}} & \overset{x_v}{\dfrac{\partial x}{\partial v}} \\ \dfrac{\partial y}{\partial u}\, y_u & \dfrac{\partial y}{\partial v}\, y_v \\ \underset{z_u}{\dfrac{\partial z}{\partial u}} & \underset{z_v}{\dfrac{\partial z}{\partial v}} \end{pmatrix}$$

$$= \int_U f\big((y_u du + y_v dv) \wedge (z_u du + z_v dv) + \dots$$

$$= \int_U f(y_u z_v - y_v z_u)\, du \wedge dv + \dots$$

$$= \int_U \Big(f(y_u z_v - y_v z_u) + g(\qquad\qquad) + h(\qquad\qquad)\, du \wedge dv$$

$$= \int_U \langle X, \varphi_u \times \varphi_v \rangle \, du \wedge dv = \int_U \langle X, \vec{n} \rangle \boxed{\|\varphi_u \times \varphi_v\| du dv}$$

$$\varphi_u \times \varphi_v = \|\varphi_u \times \varphi_v\| \cdot \vec{n}$$

Teorema di Stokes

M^{n+1} con bordo (può essere $\emptyset$) ∂M^{n+1} = n-varietà senza bordo
orientata

$\omega \in \Omega^n_c(M)$

$$\int_M d\omega = \int_{\partial M} \omega := \int_{\partial M} i^*(\omega)$$

$\mathrm{supp}(d\omega) \subseteq \mathrm{supp}(\omega)$

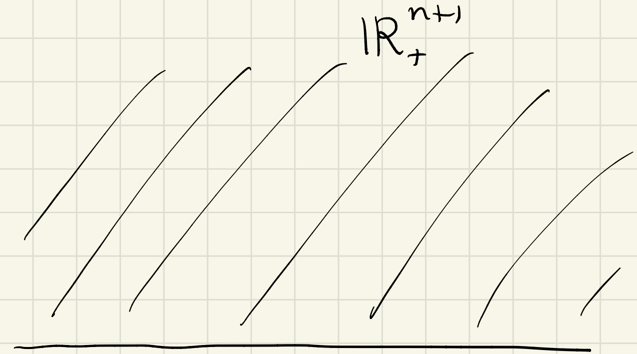
$i : \partial M \to M$ inclusione

<u>dim</u>:

Partiz. unità $\Longrightarrow$ $\omega = \omega_1 + \dots + \omega_K$ $\quad$ $\operatorname{supp} \omega_i \subseteq$ dominio di carta

Basta dimostrare per ω_i $\qquad \varphi_i : U_i \to \mathbb{R}^{n+1}_+$

" " " " $(\varphi_i^{-1})^* \omega_i$



Basta dim. per $\omega \in \Omega^n(\mathbb{R}^{n+1}_+)$

$$\omega = \sum_{i=1}^{n+1} f_i \, dx^1 \wedge \dots \wedge \widehat{dx^i} \wedge \dots \wedge dx^{n+1}$$

$$d\omega = \sum_{i=1}^{n+1} \left(\frac{\partial f_i}{\partial x_j} dx^j \right) dx^1 \wedge \dots \wedge \widehat{dx^i} \wedge \dots \wedge dx^{n+1}$$

$$= \sum_{i=1}^{n+1} \frac{\partial f_i}{\partial x_i} dx^i \wedge dx^1 \wedge \dots \wedge \widehat{dx^i} \wedge \dots \wedge dx^{n+1}$$

$$= \sum_{i=1}^{n+1} \frac{\partial f_i}{\partial x_i} (-1)^{i-1} dx^1 \wedge \dots \dots \wedge dx^{n+1}$$

Basta considerare $\omega = f dx^1 \wedge \dots \wedge \widehat{dx^i} \wedge \dots \wedge dx^{n+1}$

$$d\omega = \frac{\partial f}{\partial x^i} (-1)^{i-1} dx^1 \wedge \text{---} \wedge dx^{n+1}$$

Tesi: $\displaystyle\int_{\mathbb{R}^{n+1}_+} d\omega = \int_{\partial \mathbb{R}^{n+1}_+ = \mathbb{R}^n} \omega$

Se $1 \leq i \leq n$:

$$\int_{\mathbb{R}^{n+1}_+} d\omega = \int_{\mathbb{R}^{n+1}_+} \frac{\partial f}{\partial x^i} (-1)^{i-1} dx^1 \dots dx^{n+1}$$

$$= (-1)^{i-1} \int_{\mathbb{R}^n_+} \left(\int_{\mathbb{R}} \frac{\partial f}{\partial x^i} dx^i \right) dx^1 \text{---} \widehat{dx^i} \text{---} dx^{n+1}$$

$$\int_{\mathbb{R}} \frac{\partial f}{\partial x^i}\, dx^i = \lim_{T\to\infty} \underbrace{f(x_1,\dots,x_{i-1},T,x_{i+1},\dots,x_{n+1})}_{=0} - \underbrace{f(x_1,\dots,-T,x_i,\dots)}_{=0} = 0-0$$

$$\int_{\partial\mathbb{R}^{n+1}_+} \omega = \int f\, dx^1\wedge\dots\wedge\widehat{dx^i}\wedge\dots\wedge dx^{n+1} \qquad \text{☆}$$

$$\partial\mathbb{R}^{n+1}_+ = \mathbb{R}^n \qquad i: \partial\mathbb{R}^{n+1}_+ \to \mathbb{R}^{n+1}_+$$

$$d(\omega\wedge\eta) = d\omega\wedge\eta + (-1)^k \omega\wedge d\eta$$

$\varphi: U \to V$ qualsiasi

$$\varphi^*(\omega\wedge\eta) = (\varphi^*\omega)\wedge(\varphi^*\eta)$$

Es: $\varphi^*(d\omega) = d(\varphi^*\omega)$

$$\star := \int_{\partial \mathbb{R}^{n+1}_+} i^*\left(f\, dx^1 \wedge \dots \wedge \widehat{dx^i} \wedge \dots \wedge dx^{n+1}\right)$$

$$= \int_{\partial \mathbb{R}^{n+1}_+} f\, i^*(dx^1) \wedge \dots \wedge \widehat{dx^i} \wedge \dots \wedge \underbrace{i^*(dx^{n+1})}_{= 0}$$

Esercizio

Se $i = n+1$:

$$\omega = f\, dx^1 \wedge \dots \wedge dx^n$$

$$d\omega = (-1)^n \frac{\partial f}{\partial x^{n+1}}\, dx^1 \wedge \dots \wedge dx^{n+1}$$

$$\int_{\mathbb{R}^{n+1}_+} d\omega = (-1)^n \int_{\mathbb{R}^{n+1}_+} \frac{\partial f}{\partial x^{n+1}}\, dx^1 \dots dx^{n+1}$$

$$= (-1)^n \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^1_+ = [0,\infty)} \frac{\partial f}{\partial x^{n+1}}\, dx^{n+1} \right) dx^1 \dots dx^n$$

$$= (-1)^n \int_{\mathbb{R}^n} \Big(f(x_1, \dots, x_n, \infty) - f(x_1, \dots, x_n, 0) \Big)\, dx^1 \dots dx^n$$

$$= (-1)^{n+1} \int_{\mathbb{R}^n} f \, dx^1 \dots dx^n$$

$$\int_{\partial \mathbb{R}^{n+1}_+} \omega = \int_{\partial \mathbb{R}^{n+1}_+} f \, dx^1 \wedge \dots \wedge dx^n$$

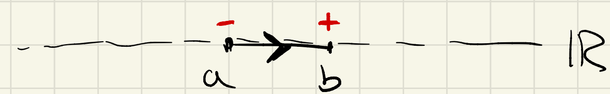
$$\left[\text{Sto usando questo fatto: } \int_M \omega = -\int_{-M} \omega \qquad \omega \in \Omega^n(M) \quad M^n \right]$$

$\square$

Conseguenze di Stokes:

$$\int_M d\omega = \int_{\partial M} \omega$$

1) $M = [a,b] \subseteq \mathbb{R}$



$\partial M = \{a,b\}$

$f = \omega \in \mathcal{C}^\infty([a,b])$

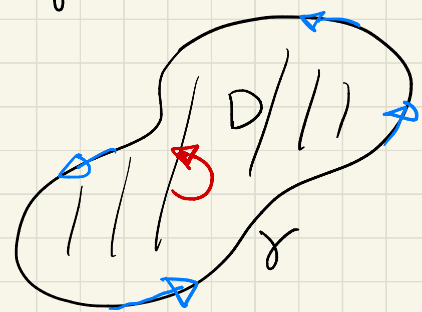
$$\int_{[a,b]} df = \int_{a \cup b} f = f(b) - f(a)$$

2)

γ in $\mathbb{R}^3$

$$\forall f \in \mathcal{C}^\infty(\mathbb{R}^3) \qquad \int_\gamma df = f(p) - f(q)$$

3) Gauss-Green



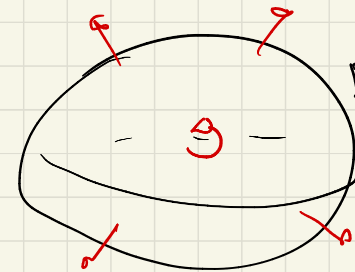
$\mathbb{R}^2$

$$\omega = f\,dx + g\,dy$$

$$d\omega = \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx \wedge dy$$

$$\int_D d\omega = \int_{\partial D = \gamma} \omega$$

4) Teorema della divergenza:

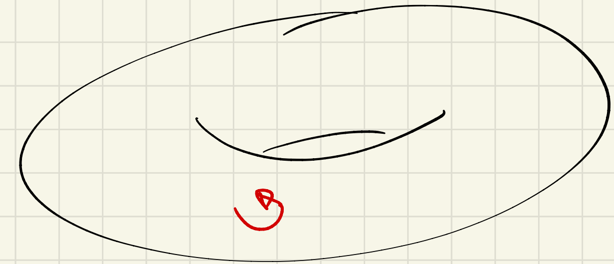


$D \subseteq \mathbb{R}^3$

$$\boxed{\int_D d\omega} = \boxed{\int_{\partial D = S} \omega} \qquad \omega \in \Omega^2(\mathbb{R}^3)$$

$$\omega = f\,dy\wedge dz + g\,dz\wedge dx + h\,dx\wedge dy$$

$$X = \begin{pmatrix} f \\ g \\ h \end{pmatrix}$$

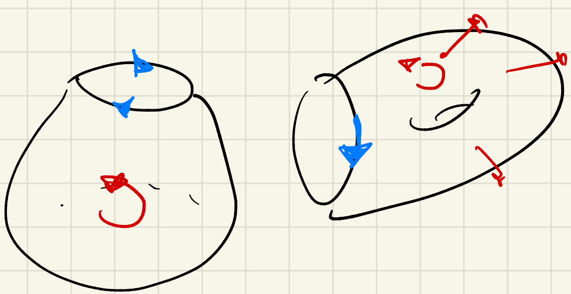


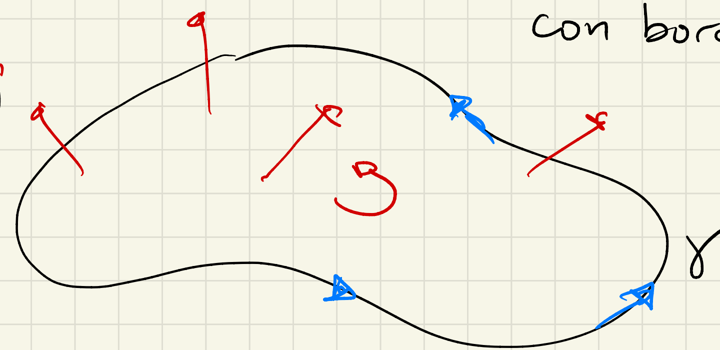
$$\star = \int_S X\cdot\vec{n}$$

$$\triangle \int_D \operatorname{div} X$$

5) Teorema di Stokes

$S \subseteq \mathbb{R}^3$ superficie con bordo





$\omega \in \Omega^1(\mathbb{R}^3) \qquad \omega = f\,dx + g\,dy + h\,dz$

$$\int_S d\omega = \int_{\partial S = \gamma} \omega = \int_\gamma X \cdot t \qquad X = \begin{pmatrix} f \\ g \\ h \end{pmatrix}$$

$$\parallel$$

$$\int_S \operatorname{rot} X \cdot \vec{n}^0 = \int_{\partial S} Y \cdot \vec{n}^0 \qquad \operatorname{rot} X = Y \text{ campo associato a } d\omega$$

<u>Oss:</u> Stokes quando $\partial M = \emptyset$

$$\int_M d\omega = \int_{\partial M = \emptyset} \omega = 0$$